International Journal of Research in Advent Technology, Vol.6, No.8, August 2018
E-ISSN: 2321-9637
Available online at www.ijrat.org

Edge – Graceful Labeling of Some Special Trees

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Abstract- In 1985, Lo [9] introduced the edge – graceful graph which is a dual notion of graceful labeling. A graph $G(V,E)$ is said to be **edge – graceful** if there exists a bijection f from E to {1, 2, …, q } such that the induced mapping f^+ from V to $\{0, 1, 2, ..., p-1\}$ given by $f^+(x) = \left(\sum f(xy)\right)(\text{mod } p)$ taken over all edges xy is a bijection. Lo [9] found a necessary condition for a graph with p vertices and q edges to be edge – graceful as $q(q+1) \equiv \frac{p(p+1)}{2}(\text{mod } p)$. Lee [7-8] conjectured that all trees of odd order are edge – graceful. In [4], Gayathri and Subbiah studied edge-graceful labeling of some trees. In this paper, we investigate the edge – graceful labeling of some special trees such as $\langle K_{1,n} : K_{1,m} \rangle$, $N\langle K_{1,n} : K_{1,m} \rangle$, Y_n, FC (1^m, $K_{1,n}$), and mG_n.

Keyword-Edge graceful labeling-EGL

1. INTRODUCTION:

Here we consider finite and undirected graphs only. For all the terms and notations we refer to Harary [5].

Labeling of graphs is an assignment of numbers to the vertices or edges or both satisfying the some constraints. If the function is defined on the vertex set it is called vertex labeling. Similarly if the function is defined on the edge set it is called an edge labeling.

By a (p, q) graph G, we mean a graph G=(V,E) with $|V|=p$ and $|E| = q$.

Labeled graphs find its applications in security measure of communication networks. Further it finds its place in X-ray crystallography and coding theory. Different applications on graph labeling can be found in [1 -2]. For an excellent survey on Graph Labeling, we refer to J. A. Gallian [3].

In 1990, Antimagic graphs was introduced by Hartsfield and Ringel [6]. A graph with q edges is said to be **antimagic** if its edges can be labeled with 1, 2, …, q such that the sums of the labels of the edges incident to each vertex are distinct.

In 1985, Lo [9] introduced the edge – graceful graph which is a dual notion of graceful labeling. Every edge – graceful graph is found to be antimagic. Lee [7-8] conjectured that all trees of odd order are edge – graceful.

By the necessary condition of edge – gracefulness of a graph one can verify that even cycles, and paths of even length are not edge – graceful. But whether trees of odd order are edge – graceful is still open. In [4], Gayathri and Subbiah studied edge-graceful labeling of some trees. In this paper, we discuss edge – graceful labeling of some special trees such as $\langle K_{1,n} : K_{1,m} \rangle$, $N\langle K_{1,n} : K_{1,m} \rangle$, Y_n, FC (1^m, $K_{1,n}$), and mG_n

2. MAIN RESULTS:

2.1 *Definition:* [9] A graph $G(V, E)$ is said to be **edge – graceful** if there exists a bijection f from E to {1, 2, ..., q} such that the induced mapping f^+ from V to {0, 1, 2, ..., p–1} given by $f^+(x) = \left(\sum f(xy)\right)(\text{mod } p)$ taken over all edges xy is a bijection.

Lo [9] found a necessary condition for a graph with p vertices and q edges to be edge – graceful as $q(q + 1) \equiv 0$ or $\frac{p}{2}$ (mod p).

2.2 *Definition:* The graph $\langle K_{1,n} : K_{1,m} \rangle$ is obtained by joining the center u of the star $K_{1,\ n}$ and the center v of another star $K_{1,\ m}$ to a new vertex w. The number of vertices is $n + m + 3$ and the number of edges is $n + m + 2$.

2.3 *Theorem:* The graph $\langle K_{1,n} : K_{1,m} \rangle$ is edge – graceful for all n, m even and $n \neq m$.

Proof: Let {w, v, v', v_1, v_2, ..., v_n, v_{n+1}, ..., v_{n+m}} be the vertices of $\langle K_{1,n} : K_{1,m} \rangle$ and {e_1, e_2, ..., e_{n+m}} ∪

$\{e, e'\}$ be the edges of $\langle K_{1,n} : K_{1,m} \rangle$ which are denoted as in Fig. 2.1.

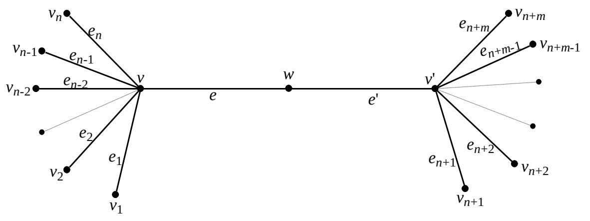

Fig. 2.1: $\langle K_{1,n} : K_{1,m} \rangle$ with ordinary labeling

We first label the edges as follows:
Consider the Diophantine equation $x_1 + x_2 = p$. The solutions are of the form $(t, p - t)$ where $1 \leq t \leq \frac{q}{2}$.

There will be $\frac{q}{2}$ pair of solutions.

We first define $f(e) = q$; $f(e') = 1$.

From other pairs, we label the edges of $K_{1,n}$ and $K_{1,m}$ by the coordinates of the pair in any order so that adjacent edges receive the coordinates of the pairs.

Then the induced vertex labels are:

$f^+(w) = 0$; $f^+(v') = 1$; $f^+(v) = q$

Now, the pendant vertices will have labels of the edges with which they are incident and they are distinct. Hence, the graph $\langle K_{1,n} : K_{1,m} \rangle$ is edge – graceful for all n, m even and $n \neq m$.

The *EGL* of $\langle K_{1,4} : K_{1,2} \rangle$ is illustrated in Fig. 2.2.

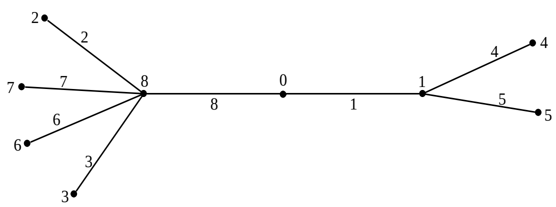

Fig. 2.2: $B_{4,2}$ with *EGL*

2.4 *Theorem:* The graph $\langle K_{1,n} : K_{1,m} \rangle$ is edge – graceful for all n, m odd and $n \neq m$.

Proof: Let $\{w, v, v', v_1, v_2, ..., v_n, v_{n+1}, ..., v_{n+m}\}$ be the vertices of $\langle K_{1,n} : K_{1,m} \rangle$ and $\{e_1, e_2, ..., e_{n+m}\} \cup \{e, e'\}$ be the edges of $\langle K_{1,n} : K_{1,m} \rangle$ which are denoted as in Fig. 2.1 of Theorem 2.3.

Consider the Diophantine equation $x_1 + x_2 = p$. The solutions are of the form $(t, p - t)$ where $1 \leq t \leq \frac{q}{2}$. There will be $\frac{q}{2}$ pair of solutions.

Among $\frac{q}{2}$ pair of solutions, choose the pair $(q, 1)$.

We now label the edges as follows:
$f(e) = q$; $f(e') = 1$; $f(e_1) = 2$; $f(e_{n+1}) = p - 2$

As we have labeled the edges e_1 and e_{n+1} there will be even number of edges incident with each v and v' to be labeled. We label these edges as was done in Theorem 2.3.

Then the induced vertex labels are:

$f^+(w) = 0$; $f^+(v) = 1$; $f^+(v') = p - 1$

The pendant vertices will now receive the labels of the edges with which they are incident and they are distinct. Hence, the graph $\langle K_{1,n} : K_{1,m} \rangle$ is edge – graceful for all n, m odd and $n \neq m$.

The *EGL* of $\langle K_{1,3} : K_{1,5} \rangle$ is illustrated in Fig. 2.3.

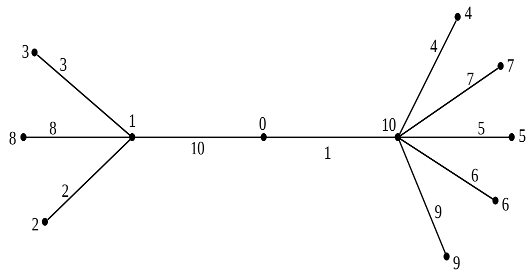

Fig. 2.3: $\langle K_{1,3} : K_{1,5} \rangle$ with *EGL*

2.5 *Theorem:* The graph $\langle K_{1,n} : K_{1,m} \rangle$ is edge – graceful for all n is even and $m = n$.

Proof: Let $\{w, v, v', v_1, v_2, ..., v_n, v_{n+1}, ..., v_{n+m}\}$ be the vertices of $\langle K_{1,n} : K_{1,m} \rangle$ and $\{e_1, e_2, ..., e_{n+m}\} \cup \{e, e'\}$ be the edges of $\langle K_{1,n} : K_{1,m} \rangle$ which are denoted as in Fig. 2.1, of Theorem 2.3.

Consider the Diophantine equation, $x_1 + x_2 = p$. The solutions are of the form $(t, p - t)$ where $1 \leq t \leq \frac{q}{2}$. There will be $\frac{q}{2}$ pair of solutions.

We first label the edges e and e^1 as follows:
$f(e) = q$; $f(e') = 1$

From the remaining pairs, label the edges of $K_{1,n}$ and $K_{1,m}$ by the coordinates of the pair in any order so that adjacent edges receives the coordinates of the pairs.

Then the induced vertex labels are:

$f^+(w) = 0$; $f^+(v') = 1$; $f^+(v) = q$

Now the pendant vertices will receive labels of the edges with which they are incident and they are distinct. Hence, the graph $\langle K_{1,n} : K_{1,m} \rangle$ is edge – graceful for all n is even.

The *EGL* of $\langle K_{1,4} : K_{1,4} \rangle$ is illustrated in Fig. 2.4.

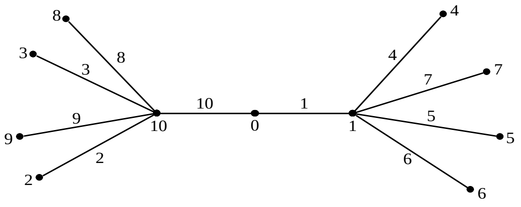

Fig. 2.4: $\langle \boldsymbol{K_{1,4} : K_{1,4}} \rangle$ with *EGL*

2.6 ***Theorem:*** The graph $\langle K_{1,n} : K_{1,n} \rangle$ is edge – graceful for all n is odd.

Proof: Let $\{w, v, v', v_1, v_2, ..., v_n, v_{n+1}, ..., v_{n+m}\}$ be the vertices of $\langle K_{1,n} : K_{1,m} \rangle$ and $\{e_1, e_2, ..., e_{n+m}\} \cup \{e, e'\}$ be the edges of $\langle K_{1,n} : K_{1,m} \rangle$ which are denoted as in Fig. 2.1, of Theorem 2.3.

Consider the Diophantine equation $x_1 + x_2 = p$. The solutions are of the form $(t, p - t)$ where $1 \le t \le q/2$. There will be $q/2$ pair of solutions.

We first label the edges as follows:

$f(e) = q; \qquad f(e') = 1$

$f(e_1) = 2; \qquad f(e_{n+1}) = p - 2$

As we have labeled the edges e_1 and e_{n+1}, there will be even number of edges incident with each v and v^1 to be labeled. We label these edges as was done in Theorem 2.3.

Then the induced vertex labels are:

$f^+(w) = 0; \quad f^+(v) = 1; \quad f^+(v') = p - 1$

The pendant vertices will have the labels of the edges with which they are incident and they are distinct. Hence, the graph $\langle K_{1,n} : K_{1,n} \rangle$ is edge – graceful for all n is odd.

The *EGL* of $\langle K_{1,3} : K_{1,3} \rangle$ is illustrated in Fig. 2.5.

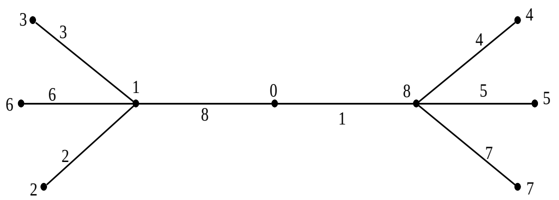

Fig. 2.5: $\langle \boldsymbol{K_{1,3} : K_{1,3}} \rangle$ **with** ***EGL***

2.7 ***Theorem:*** The graph $\langle K_{1,n} : K_{1,m} \rangle$ of even order is not an edge – graceful graph.

Proof: The graph $\langle K_{1,n} : K_{1,m} \rangle$ will be of even order in the following cases.

(i) n is odd and m is even

(ii) n is even and m is odd

In both cases, $p = n + m + 3$ which is even and hence by Lo's necessary condition, it is not edge – graceful.

2.8 ***Definition:*** The graph $N\langle K_{1,n} : K_{1,m} \rangle$ is obtained by joining the each center of N copies of $\langle K_{1,n} : K_{1,m} \rangle$ to a new vertex w.

2.9 ***Theorem:*** The graph $N\langle K_{1,n} : K_{1,m} \rangle$ $(n, m \ge 2)$ is edge – graceful for all N, n, m is even.

Proof: Let $\{u_i, v_i, w_i : 1 \le i \le N\} \cup \{w_0\} \cup \{u_{ij} : 1 \le i \le N, 1 \le j \le n\} \cup \{v_{i,j} : 1 \le i \le N, 1 \le j \le m\}$ be the vertices and $\{a_{i1}, a_{i2} : 1 \le i \le N\} \cup \{e_i : 1 \le i \le N\} \cup \{e_{ij} : 1 \le i \le N, 1 \le j \le n\} \cup \{g_{ij} : 1 \le j \le N, 1 \le j \le m\}$ be the edges of $N\langle K_{1,n} : K_{1,m} \rangle$ which are denoted as in Fig. 2.6.

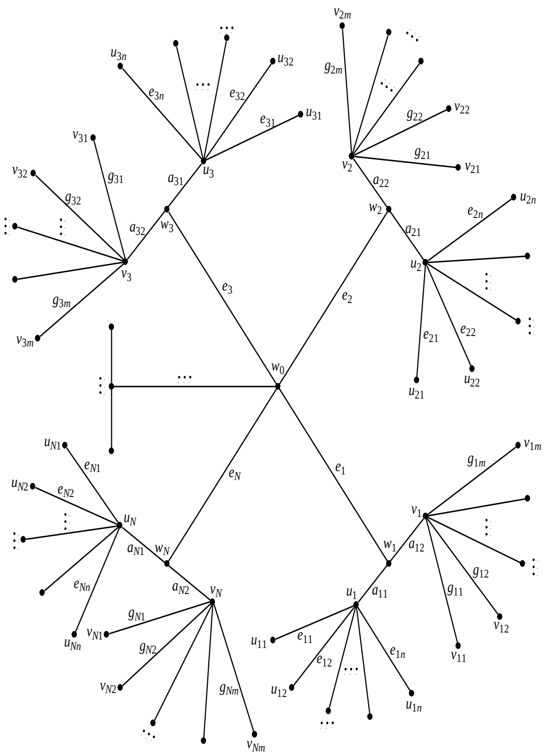

Fig. 2.6: $N\langle \boldsymbol{K_{1,n} : K_{1,m}} \rangle$ with ordinary labeling

Consider the Diophantine equation $x_1 + x_2 = p$. The pair of solutions are $(t, p - t)$ where $1 \le t \le \frac{q}{2}$. There are $\frac{q}{2}$ pair of solutions. With these pairs, we label the edges by the co-ordinates of the pair in any order so that the adjacent edges receive the coordinates of the pairs. Now the pendant vertices will have the labels of the edges with which they are incident and they are distinct.

Now the induced vertex labels are follows:

$$f^+(w_i) = w_i; \quad 1 \le i \le N;$$

$$f^+(w_0) = 0.$$

$$f^+(w_i) = f(e_i); \ 1 \le i \le N$$

$$f^+(u_i) = f^+(a_{i1}); \ 1 \le i \le N$$

$$f^+(v_i) = f^+(a_{i2}); \ 1 \le i \le N$$

Hence, the graph $N\langle K_{1,n} : K_{1,m}\rangle$ is $(n, m \ge 2)$ is edge – graceful for all N, n, m is even.

The *EGL* of $4\langle K_{1,2} : K_{1,2}\rangle$ and $4\langle K_{1,2} : K_{1,4}\rangle$ are illustrated in Fig. 2.7 and Fig. 2.8 respectively.

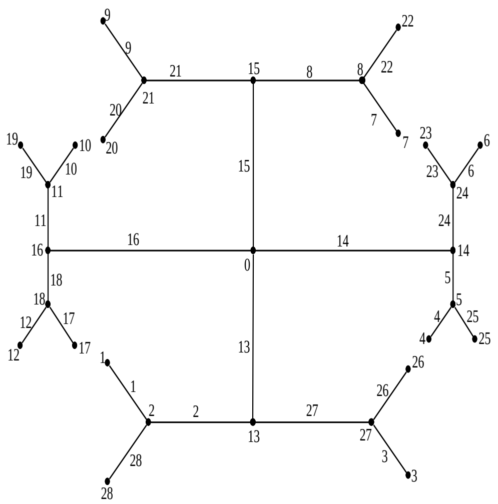

Fig. 2.7: $\mathbf{4\langle K_{1,2} : K_{1,2}\rangle}$ with *EGL*

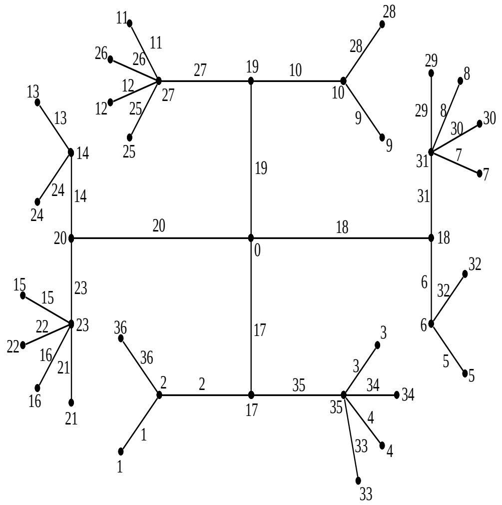

Fig. 2.8: $\mathbf{4\langle K_{1,2} : K_{1,4}\rangle}$ with *EGL*

2.10 ***Definition:*** A ***y*-tree** is a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point and it is denoted by Y_n where n is the number of vertices in the tree.

2.11 ***Theorem:*** The y-tree Y_n is edge – graceful for all n odd and $n \ge 11$.

Proof: Let $\{v_1, v_2, ..., v_n\}$ be the vertices of Y_n and $\{e_1, e_2, ..., e_{n-1}\}$ be the edges of Y_n which are denoted as in Fig. 2.9.

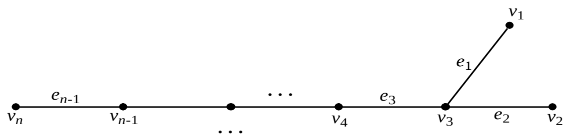

Fig. 2.9: Y_n with ordinary labeling

We first label the edges as follows:

$$f(e_i) = i; \ 1 \le i \le \frac{n+1}{2}$$

$$f\left(e_{\frac{n+3}{2}}\right) = \frac{n+5}{2}$$

$$f\left(e_{\frac{n+5}{2}}\right) = \frac{n+3}{2}$$

$$f(e_i) = I; \ \frac{n+7}{2} \le i \le n-1$$

Then the induced vertex labels are:

$$f^+(v_i) = I; \ 1 \le i \le 2$$

$$f^+(v_3) = 6$$

$$f^+(v_i) = 2i - 1;\ 4 \le i \le \frac{n-1}{2}$$

$$f^+\left(v_{\frac{n+1}{2}}\right) = 0$$

$$f^+(v_i) = \frac{3+2i-n}{2};\ \frac{n+3}{2} \le i \le \frac{n+7}{2}$$

$$f^+(v_i) = 2i - n - 1;\ \frac{n+9}{2} \le i \le n$$

Clearly, the vertex labels are all distinct. Hence, the *y*-tree Y_n is edge – graceful for all *n* odd and $n \ge 11$.

The *EGL* of Y_9 and Y_{13} are illustrated in Fig. 2.10 and Fig. 2.11 respectively.

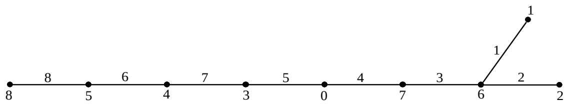

Fig. 2.10: Y_9 with *EGL*

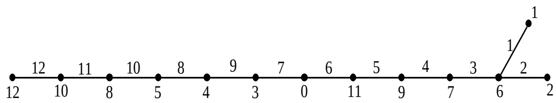

Fig. 2.11: Y_{13} with *EGL*

2.12 ***Definition*:** A fire – cracker is a graph obtained from the concatenation of *m* copies of the stars $K_{1,n}$ by linking one leaf from each. It is denoted by $FC(1^m, K_{1,n})$.

2.13 ***Theorem*:** The graph $FC(1^m, K_{1,n})$ is edge – graceful for all *m*, *n* even.

Proof: Let $\{w, w_1, w_2, ..., w_m\} \cup \{u_{ij} : 1 \le i \le m, 1 \le j \le n\}$ be the vertices and $\{f_i : 1 \le i \le m\} \cup \{e_{ij} : 1 \le i \le m, 1 \le j \le n\}$ be the edges of $FC(1^m, K_{1,n})$ which are denoted as in Fig. 2.12.

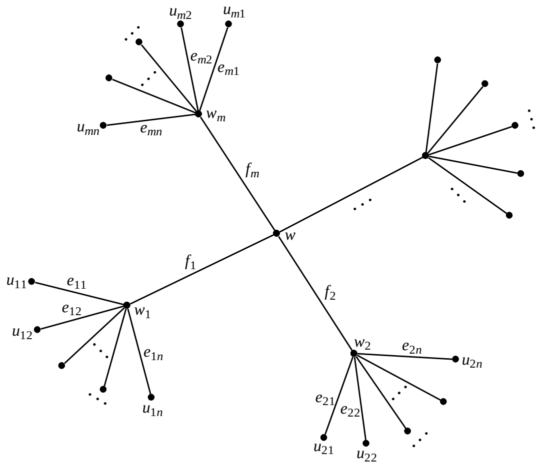

Fig. 2.12: $FC(1^m, K_{1,n})$ with ordinary labeling

Consider the Diophantine equation $x_1 + x_2 = p$. The solutions are of the form $(t, p - t)$ where $1 \le t \le \frac{q}{2}$. There will be $\frac{q}{2}$ pair of solutions. With these pairs, we label the edges of $FC(1^m, K_{1,n})$ by the coordinates of the pairs, in any order so that adjacent edges receive the coordinates of the pairs. Then the induced vertex label are $f^+(w) = 0$; $f^+(w_i) = f^+(f_i)$ $1 \le i \le m$ and the pendant vertices will have labels of the edges with which they are incident and they are distinct. Hence, the graph $FC(1^m, K_{1,n})$ is edge – graceful for all *m*, *n* even.

The *EGL* of $FC(1^4, K_{1,4})$ and $FC(1^4, K_{1,6})$ are illustrated in Fig. 2.13 and Fig. 2.14 respectively.

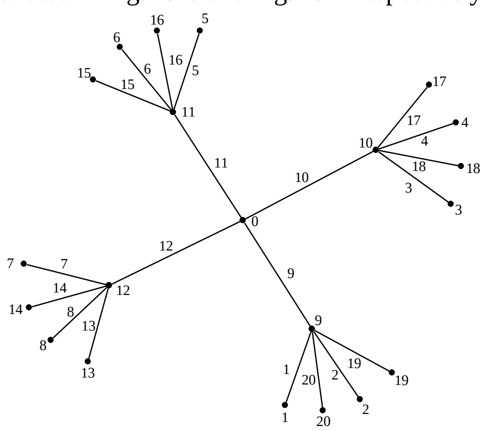

Fig. 2.13: $FC(1^4, K_{1,4})$ with *EGL*

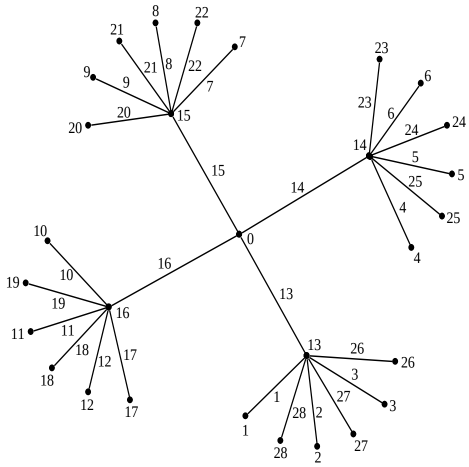

Fig. 2.14: $FC(1^4, K_{1,6})$ with *EGL*

2.14 *Definition:* The graph mG_n is obtained from m copies of $\langle K_{1,n} : K_{1,n} \rangle$ by joining one leaf of i^{th} copy of $\langle K_{1,n} : K_{1,n} \rangle$ with the center of $(i + 1)^{\text{th}}$ copy of $\langle K_{1,n} : K_{1,n} \rangle$ where $1 \le i \le m - 1$.

2.15 *Theorem:* The graph mG_n is edge – graceful for n is even and for all m.

Proof: Let $\{u_i : 1 \le i \le m\} \cup \{v_i : 1 \le i \le m\} \cup \{w_i : 1 \le i \le m\} \cup \{u_{ij}, v_{ij} : 1 \le i \le m, 1 \le j \le n\}$ be the vertices and $\{e_{ij}, g_{ij} : 1 \le i \le m, 1 \le j \le n\} \cup \{a_{i1}, a_{i2} : 1 \le i \le m\}$ be the edges of mG_n which are denoted as in Fig. 2.15.

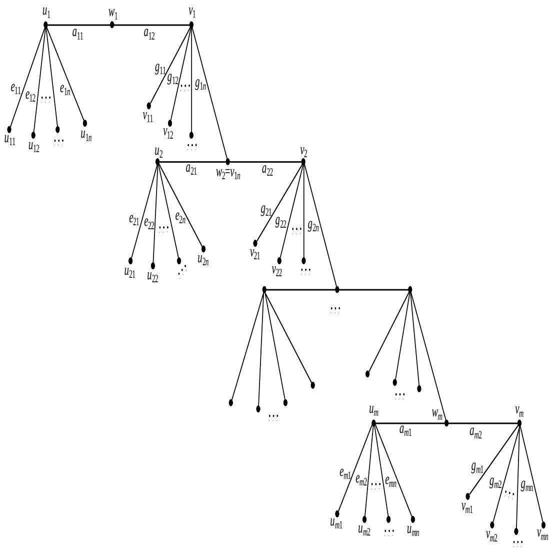

Fig. 15: mG_n with ordinary labeling

Consider the Diophantine equation $x_1 + x_2 = p$. The solutions are of the form $(t, p - t)$, where $1 \le t \le \frac{q}{2}$. There are $\frac{q}{2}$ pair of solutions. With these pairs, we label the edges by the coordinates of the pair in any order so that the adjacent edges receive the coordinates of the pairs.

Then the induced vertex labels are:

$$f^+(u_i) = f(a_{i1}); \qquad 1 \le i \le m$$

$$f^+(v_i) = f(a_{i2}); 1 \le i \le m$$

$$f^+(w_1) = 0$$

$$f^+(w_i) = f\left(g_{(i-1)n}\right); \qquad 2 \le i \le m$$

And all the pendant vertices will have the labels of the edges with which they are incident and are distinct.

Hence, the graph mG_n is edge – graceful for n is even and for all m.

The *EGL* of $3G_2$ and $2G_4$ are illustrated in Fig. 2.16 and Fig. 2.17 respectively.

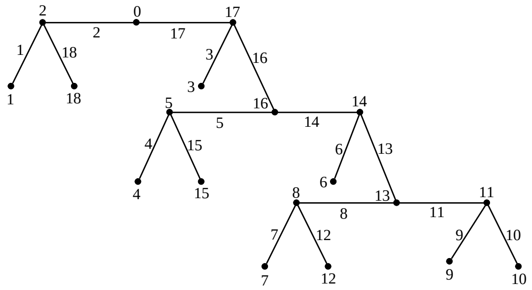

Fig. 2.16: $3G_2$ with *EGL*

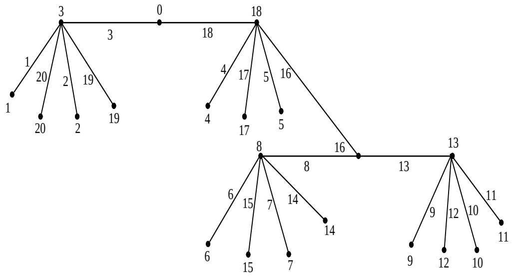

Fig. 2.17: $2G_4$ with *EGL*

3. CONCLUSION

In this paper, we investigated, edge-graceful labeling of some special trees, which lead to the Lo's conjecture in affirmative. Similar such trees can be examined for edge-gracefulness towards attempting to the conjecture.

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